Chaotic oscillations in nonlinear system of interacting oscillators with the interaction of the fourth order.

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Abstract

Dynamics of two anharmonic oscillators with interaction of the fourth order has been investigated. The conditions at realization of which system is integrable are established. The exact analytical solution of the nonlinear equations in the case of adiabatic isolation of a system of oscillators has been obtained.

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In optics of weak light streams the polarization of an electromagnetic wave in substance does not depend on intensity of light and univocally corresponds to the polarization of emanation, impinging on the boundary of medium - vacuum. The situation cardinally varies in the case of nonlinear optics, when the factors of a refraction and absorption of substance become functions of intensity of a radiation [1]. At a particular combination of parameters of a radiation and nonlinear medium, the stationary polarization of light " on an inlet " becomes unstable " on an exit ", or there can be a random modification of polarization in time [2-7]. The specified phenomenon is called as a spontaneous violation of polarization symmetry.

In a strong light field, when the nonlinear component of the electrical induction becomes noticeable with respect to the linear component, traditional exposition based on a material equation of a medium, which represents a series with respect of terms of electric field is impossible. In works [8-9] the prospects of observation of amplitude instability for mediums in which nonlinear response is featured within the framework of a model of the Duffing oscillator were considered. In work [10] the nonlinear susceptibility of the ensemble of chaotically oriented mirror - asymmetrical molecules has been estimated. Assumption that each of molecules can be represented by the nonlinear system of interacting oscillators in the field of electromagnetic wave has been done:

$$\ddot{x}+\omega_0^2 x+\alpha y+\Gamma\dot{x}+ax^3+b(y^3+3x^2y)+cxy^2=\frac{eE_z}{m}\exp(iK_z\frac{D}{2})$$
$$\ddot{x}+\omega_0^2 y+\alpha x+\Gamma\dot{y}+ay^3+b(x^3+3y^2x)+cyx^2=\frac{eE_y}{m}\exp(iK_z\frac{D}{2}) \qquad (1)$$

Where ω_0 is a natural frequency of small oscillations of oscillators creating a molecule, e and m electrical charge and mass accordingly, E_z, E_y projections of amplitude of an electric field, D is a distance between oscillators, a , b , c are the anharmonic constants naturally of oscillators and connection between them, Γ is the parameter describing attenuation. With the help of summation of the dipole moment of molecules

$$d_{x,y}=e\begin{pmatrix}x\\y\end{pmatrix}\exp(\pm iK_z\frac{D}{2})$$

and averaging with respect of small volume, in the work [10], the expression for macroscopic polarization was obtained, and was shown that in mediums with nonlinear gyrotropy the polarization instability takes place.

Despite of it, due to the great importance for the nonlinear optics, the immediate research of the system of nonlinear oscillators (1) is of interest.

The purpose of given work is the qualitative, theoretical research of the equations (1) and establishment of relations between parameters of a system at which equation (1) will be integrable.

Let us consider at first the case of weak nonlinearity. We also suppose that the amplitude of the external field is small:

$$\omega_0^2 >> \frac{eE_z}{m}$$

Considering terms with cubic nonlinearity as a perturbation, let us rewrite system (1) in following form:

$$\begin{aligned} \ddot{x} + \omega_0^2 x + \alpha y &= -\mu f(x, \dot{x}, y, \dot{y}), \\ \ddot{y} + \omega_0^2 y + \alpha x &= -\mu g(x, \dot{x}, y, \dot{y}), \end{aligned} \tag{2}$$

where

$$\begin{aligned} f(x, \dot{x}, y, \dot{y}) &= x^3 + y^3 + 3x^2 y + xy^2 + \frac{\Gamma}{\mu}\dot{x} \\ g(x, \dot{x}, y, \dot{y}) &= x^3 + y^3 + 3xy^2 + x^2 y + \frac{\Gamma}{\mu}\dot{y} \end{aligned} \tag{3}$$

here are used the following designations:

$$x \to a_0 x,\ y \to a_0 y,\ \omega_0 \to \omega_0\sqrt{\frac{ma_0}{eE_{z,y}}},\ \Gamma \to \Gamma\sqrt{\frac{ma_0}{eE_{z,y}}},\ \alpha \to \alpha\frac{ma_0}{eE_{z,y}},\ t \to t\sqrt{\frac{eE_{z,y}}{ma_0}}$$

$$\mu \approx a\frac{ma_0^3}{eE_{z,y}} \approx b\frac{ma_0^3}{eE_{z,y}} \approx c\frac{ma_0^3}{eE_{z,y}},$$

and transformation to the dimensionless variables and parameters (where - a_0 means a Bohr radius) has been done. For $\mu = 0$ the set of equations (2) has the solution:

$$\begin{aligned} x &= \alpha_1 \sin(k_1 t + \beta_1) + \alpha_2 \sin(k_2 t + \beta_2) \\ y &= -\alpha_1 \sin(k_1 t + \beta_1) + \alpha_2 \sin(k_2 t + \beta_2) \end{aligned} \tag{4}$$

Here α_1, α_2, β_1, β_2 are the constants of integration, $k_1 = \sqrt{\omega_0^2 - \alpha_1}$, $k_2 = \sqrt{\omega_0^2 - \alpha_2}$

Solution of the set of equations (2) for $\mu \neq 0$ we shall search in the form (4), considering that $\alpha_1, \alpha_2, \beta_1, \beta_2$ are slowly varying functions of time. The additional conditions which we shall impose on the functions $\alpha_1, \alpha_2, \beta_1, \beta_2$ for their determinancy consist in the following: first order temporary derivatives of x and y are having the same form, as in the case of constant $\alpha_1, \alpha_2, \beta_1, \beta_2$. As a result we get:

$$\begin{aligned} \dot{\alpha}_1\ sin(k_1 t + \beta_1) + \dot{\alpha}_2\ \sin(k_2 t + \beta_2) + \alpha_1\dot{\beta}_1\ \cos(k_1 t + \beta_1) + \alpha_2\dot{\beta}_2\ \cos(k_2 t + \beta_2) &= 0 \\ -\dot{\alpha}_1\ \sin(k_1 t + \beta_1) + \dot{\alpha}_2\ \sin(k_2 t + \beta_2) - \alpha_1\dot{\beta}_1\ \cos(k_1 t + \beta_1) + \alpha_2\dot{\beta}_2\ \cos(k_2 t + \beta_2) &= 0 \end{aligned} \tag{5}$$

Calculating temporary derivatives of $\dot{x}$, and $\dot{y}$ and substituting obtained expressions in the equations (2) one can get:

$$\begin{aligned} \dot{\alpha}_1 k_1 \cos(k_1 t + \beta_1) + \dot{\alpha}_2 k_2 \cos(k_2 t + \beta_2) - k_1\alpha_1\dot{\beta}_1\ \sin(k_1 t + \beta_1) + k_2\alpha_2\dot{\beta}_2\ \sin(k_2 t + \beta_2) &= \mu f* \\ -\dot{\alpha}_1 k_1 cos(k_1 t + \beta_1) + \dot{\alpha}_2 k_2 \cos(k_2 t + \beta_2) - k_1\alpha_1\dot{\beta}_1\ \sin(k_1 t + \beta_1) - k_2\alpha_2\dot{\beta}_2\ \sin(k_2 t + \beta_2) &= \mu g* \end{aligned} \tag{6}$$

Where

$$
\begin{aligned}
&f^* = f[\alpha_1 \sin(k_1 t + \beta_1) + \alpha_2 \sin(k_2 t + \beta_2)\,; \alpha_1 k_1 \cos(k_1 t + \beta_1) + \alpha_2 k_2 \cos(k_2 t + \beta_2)\,; \\
&\quad -\alpha_1 \sin(k_1 t + \beta_1) + \alpha_2 \sin(k_2 t + \beta_2)\,;\ -\alpha_1 k_1 \cos(k_1 t + \beta_1) + \alpha_2 k_2 \cos(k_2 t + \beta_2)\ ], \\
&g^* = g[\alpha_1 \sin(k_1 t + \beta_1) + \alpha_2 \sin(k_2 t + \beta_2)\,; \alpha_1 k_1 \cos(k_1 t + \beta_1) + \alpha_2 k_2 \cos(k_2 t + \beta_2)\,; \\
&\quad -\alpha_1 \sin(k_1 t + \beta_1) + \alpha_2 \sin(k_2 t + \beta_2)\,;\ -\alpha_1 k_1 \cos(k_1 t + \beta_1) + \alpha_2 k_2 \cos(k_2 t + \beta_2)\ ]
\end{aligned} \tag{7}
$$

The equations (5), (6), (7) are a set of equations for the determination of $\alpha_1\,, \alpha_2\,, \beta_1\,, \beta_2$. After solving of these equations with respect to $\dot{\alpha}_1$one can obtain:

$$
\frac{d\alpha_1}{dt} = \frac{\Delta_1}{\Delta}
$$

$$
\Delta = \begin{vmatrix} \sin\xi & \sin\eta & \cos\xi & \cos\eta \\ -\sin\xi & \sin\eta & -\cos\xi & \cos\eta \\ k_1\cos\xi & k_2\cos\eta & -k_1\sin\xi & -k_2 sin\eta \\ -k_1\cos\xi & k_2\cos\eta & k_1\sin\xi & -k_2\sin\eta \end{vmatrix},
$$

$$
\Delta_1 = \begin{vmatrix} 0 & \sin\eta & \cos\xi & \cos\eta \\ 0 & \sin\eta & -\cos\xi & \cos\eta \\ \mu f^* & k_2\cos\eta & -k_1\sin\xi & -k_2 sin\eta \\ \mu g* & k_2\cos\eta & k_1\sin\xi & -k_2\sin\eta \end{vmatrix},
$$

here $\xi = k_1 t + \beta_1$and $\eta = k_2 t + \beta_2$.
Making similar operations with respect to $\alpha_1\,, \alpha_2\,, \beta_1\,, \beta_2$ after unwieldy calculations we get:

$$
\begin{aligned}
\frac{d\alpha_1}{dt} &= \frac{\mu}{k_1(k_2^2 - k_1^2)}[\alpha f^* - \alpha g^*]\ \cos\xi, \\
\frac{d\alpha_2}{dt} &= \frac{\mu}{k_2(k_2^2 - k_1^2)}[\alpha f^* + \alpha g^*]\ \cos\eta, \\
\alpha_1\frac{d\beta_1}{dt} &= \frac{\mu}{k_1(k_2^2 - k_1^2)}[\alpha f^* - \alpha g^*]\ \sin\xi, \\
\alpha_2\frac{d\beta_2}{dt} &= -\frac{\mu}{k_1(k_2^2 - k_1^2)}[\alpha f^* + \alpha g^*]\ \sin\eta.
\end{aligned} \tag{8}
$$

The obtained equations (8) represent a set of equations (2), transformed to another variables. Let's assume, variation of the $\alpha_1\,, \alpha_2\,, \beta_1\,, \beta_2$ are slow than oscillations in the initial dynamical system. Averaging the obtained equations with respect to phase $\frac{2\pi}{k_1}$ and $\frac{2\pi}{k_2}$ for $\alpha_1\,, \alpha_2\,, \beta_1\,, \beta_2$ we shall get:

$$
\begin{aligned}
\dot{\alpha}_1 &= \frac{\mu}{2\alpha\sqrt{\omega_0^2 - \alpha}}(F_1 - G_1), \\
\dot{\alpha}_2 &= \frac{\mu}{2\alpha\sqrt{\omega_0^2 + \alpha}}(F_2 + G_2), \\
\dot{\beta}_1 &= -\frac{\mu}{2\alpha\sqrt{\omega_0^2 - \alpha}}(F_3 - G_3), \\
\dot{\beta}_2 &= -\frac{\mu}{2\alpha\sqrt{\omega_0^2 + \alpha}}(F_4 + G_4).
\end{aligned} \tag{9}
$$

Here

$$
\begin{aligned}
F_1 &= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} f^*\cos\xi d\xi d\eta;\, G_1 = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} g^*\cos\xi d\xi d\eta;\\
F_2 &= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} f^*\cos\eta d\xi d\eta;\, G_2 = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} g^*\cos\eta d\xi d\eta;\\
F_3 &= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} f^*\sin\xi d\xi d\eta;\, G_3 = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} g^*\sin\xi d\xi d\eta;\\
F_4 &= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} f^*\sin\eta d\xi d\eta;\, G_4 = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} g^*\sin\eta d\xi d\eta;
\end{aligned}
\tag{10}
$$

The right sides of the equations (8) do not depend on β_1 and β_2. Therefore they serve for the investigation of α_1 and α_2. Substituting (7) in (10) and integrating finally we get:

$$
\alpha_1(t) = \alpha_1(0)e^{-\frac{\Gamma}{\alpha(\omega_0^2-\alpha)^{1/2}}t}\,, \quad \alpha_1(t) = \alpha_1(0)e^{-\frac{\Gamma}{\alpha(\omega_0^2+\alpha)^{1/2}}t}. \tag{11}
$$

It is easy to see from (11) that in case of weak nonlinearity, the nonlinear terms does not affect on the form of motion. Therefore the application of the method of slowly varying amplitudes in case of strong nonlinearity made in [9] is incorrect. As a result in the system originates damping oscillations and the phase diagram represents a steady focal point. This outcome can be considered as a singularity of the system (1). These reasonings are proved also by numerical calculations (see Fig. 1).

Fig. 1 Phase diagram obtained by numerical calculations for the values of the dimensionless parameters: $\omega_0^2 = 10\,,\ \alpha = 1,\ \Gamma = 0,5\,,\ f = 0,1\,,\ a = 0,6\,,\ \ b = 0,5\,,\ c = 0,5$

As show numerical calculations at the action on the system of a strong exterior field

$$
\omega_0^2 = 0.1\frac{eE_{x,y}}{ma_0} \tag{12}
$$

in the phase diagram there is a limit cycle [11-13] (see Fig. 2).

Fig. 2 Phase diagram obtained by numerical calculations for the values of the dimensionless parameters:

$\omega_0^2 = 10, \alpha = 1, \Gamma = 0.5,\ \ f = 1.7\,,\ a = 0.6\ \ \ b = 0,5\,,\ \ c = 0.5.$

In case of strong nonlinearity the application of the method of slowly varying amplitudes becomes impossible. However below will be shown that under particular conditions is possible to obtain analytical solutions of the set of equations (1) even in the presence of interaction terms of the fourth order.

It is easy to see that in case of absence external field and damping, (i.e. in case of adiabatic apartness) the equations (1) can be generated by the Hamiltonian

$$
H = P_x^2/2 + P_y^2/2 + \omega_0^2/2\,(x^2+y^2) + \alpha xy + a/4(x^4+y^4) + c/2\,x^2y^2 + b(xy^3+x^3y) \tag{13}
$$

After the transformation to the variables $q_1 = x + y\,, q_2 = x - y$ and supposing that $c = 3a$ one can obtain:

$$
\begin{aligned}
H &= H_1 + H_2,\\
H_1 &= \frac{p_1^2}{2} + \frac{a+b}{8}q_1^4 + \frac{1}{4}(\omega_0^2+\alpha)q_1^2,\\
H_2 &= \frac{p_2^2}{2} + \frac{a+b}{8}q_2^4 + \frac{1}{4}(\omega_0^2-\alpha)q_2^2.
\end{aligned}
\tag{14}
$$

The similar result can be obtained with the using of more general method (the method offered by P.Lax [14]). This method allows solving the problem even in case when determinations of new variables are not so trivial.

Let's initiate the analysis of the Hamiltonian (13). The corresponding equations look like

$$\ddot{q}_{1,2} = -\frac{a+b}{2} q_{1,2}^3 - \frac{\omega_0^2 \pm \alpha}{2} q_{1,2} \tag{15}$$

Representing $\ddot{q}_{1,2}$ as $\ddot{q}_{1,\,2} = \frac{1}{2\dot{q}_{1,\,2}} \frac{d}{dt} (\dot{q}_{1,\,2})^2$and substituting in (15) we get:

$$\dot{q}_{1,2} = \frac{\sqrt{|a+b|}}{2} \sqrt{((q_{1,2}^2 - \phi_{1,2}^2)(q_{1,2}^2 - \Phi_{1,2}^2))} \tag{16}$$

where

$$\theta_{1,2} = \dot{q}_{1,2}(0) + \frac{(a+b)}{4} (q_{1,2}(0))^4 + \frac{\omega_0^2 \pm \alpha}{2} (q_{1,2}(0))^2,$$

$$\phi_{1,2} = \frac{\omega_0^2 \pm \alpha + \sqrt{(\omega_0^2 \pm \alpha)^2 - 4\theta_{1,\,2}\,|a+b|}}{|a+b|},$$

$$\Phi_{1,2} = \frac{\omega_0^2 \pm \alpha - \sqrt{(\omega_0^2 \pm \alpha)^2 - 4\theta_{1,\,2}\,|a+b|}}{|a+b|}.$$

In (16) we have made assumption that $a+b<0$ because integration of (17) can be done only in this case. The integration of (16) gives

$$\frac{1}{\Phi_{1,2}} F(\arcsin \frac{q_{1,2}(t)}{\phi_{1,2}} ; \frac{\phi_{1,2}^2}{\Phi_{1,2}^2}) = \frac{\sqrt{|a+b|}t}{2} \tag{17}$$

where F(...) is the elliptic integral of the first kind [15]. Having produced a reversion of the expression (17) finally we get:

$$q_{1,2}(t) = \phi_{1,2} sn(\frac{\sqrt{|a+b|}}{2} \Phi_{1,2} t \, ; \frac{\phi_{1,2}^2}{\Phi_{1,2}^2}) \tag{18}$$

where sn (....) is the elliptic sine of the Jacobi [15].

The expressions (18) are general solutions of the equations corresponding to the Hamiltonian (13). By means of inverse transformations $x = \frac{q_1+q_2}{2}$; $y = \frac{q_1-q_2}{2}$ they allow to express solution of the initial problem in the analytical form through higher transcendental functions. In case of the initial conditions $\theta_{1,2} = \frac{1}{4} \frac{(\omega_0^2 \pm \alpha)^2}{|a+b|}$

the solutions become considerably simpler [15] and takes form of solitary waves [16]

$$x = \frac{1}{2} (\sqrt{\frac{\omega_0^2 + \alpha}{|a+b|}} \tanh \sqrt{\frac{\omega_0^2 + \alpha}{2}} t + \sqrt{\frac{\omega_0^2 - \alpha}{|a+b|}} \tanh \sqrt{\frac{\omega_0^2 - \alpha}{2}} t)$$
$$y = \frac{1}{2} (\sqrt{\frac{\omega_0^2 + \alpha}{|a+b|}} \tanh \sqrt{\frac{\omega_0^2 + \alpha}{2}} t - \sqrt{\frac{\omega_0^2 - \alpha}{|a+b|}} \tanh \sqrt{\frac{\omega_0^2 - \alpha}{2}} t) \tag{19}$$

For obtaining of solutions (20), (21) we neglected a dissipation of energy and influence of external field, i.e. we suppose that the system (1) is adiabatic isolated and change of energy during the one period of oscillations can be neglected. Taking into account that the period of the solution (18) is defined by the expression

$$T = 4K(\frac{\phi^2_{1,2}}{\Phi^2_{1,2}})$$

(where K (...) is the complete elliptic integral [15]), it is possible to announce that expressions (18) and (19) are valid for the intervals of time

$$t < \frac{\alpha\sqrt{(\omega_0^2 \pm \alpha)}}{\Gamma} 4K(\frac{\phi^2_{1,2}}{\Phi^2_{1,2}}) \tag{20}$$

In case of intervals of time $t > \frac{\alpha\sqrt{(\omega_0^2 \pm \alpha)}}{\Gamma} T$, under the action on the system of external constant field the obtaining of the analytical solution is impossible and it is necessary to use numerical methods.

In case of strong nonlinearity for the particular values of parameters in the system can occur chaotic oscillations [11-13] (see Fig 3, Fig 4, Fig 5).

Fig. 3 $x(t)$ as a function of time obtained by numerical calculations for the values of the dimensionless parameters: $\omega_0^2 = 10, \quad \alpha = 1, \quad \Gamma = 0.1, \quad f = 1.7\,, \quad a = 3, \quad b = 2.5\,, \quad c = 2.5$

Fig. 4 as a function of time obtained by numerical calculations for the values of the dimensionless parameters: $\omega_0^2 = 10, \quad \alpha = 1.2, \quad \Gamma = 0.1, \quad f = 1.5\,, \quad a = 3, \quad b = 2.5\,, \quad c = 2.5$

Fig. 5 Projection of trajectory of phase point motion on the space $x(t)$, $dx(t)/dt$, $y(t)$ for the values of parameters: $\omega_0^2 = 10, \quad \alpha = 1, \quad \Gamma = 0.1, \quad f = 1.7\,, \quad a = 3, \quad b = 2.5\,, \quad c = 2.5$ obtained by numerical calculations.

It is ease to see from Fig. 3, Fig. 4 and Fig. 5 that motion is realized in the restricted region of phase space. The external field compensates energy dissipation and we have not focal points. On the other hand motion is not periodically and as a result trajectory is not closed (see Fig.5). In another words it means that ratios between frequencies of oscillations of $x(t)$, $\dot{x}(t)$, $y(t)$ are not rational numbers. So in case of strong nonlinearity, under the action of external constant field in the system occurs chaotic non-periodical oscillations and system is not integrable.

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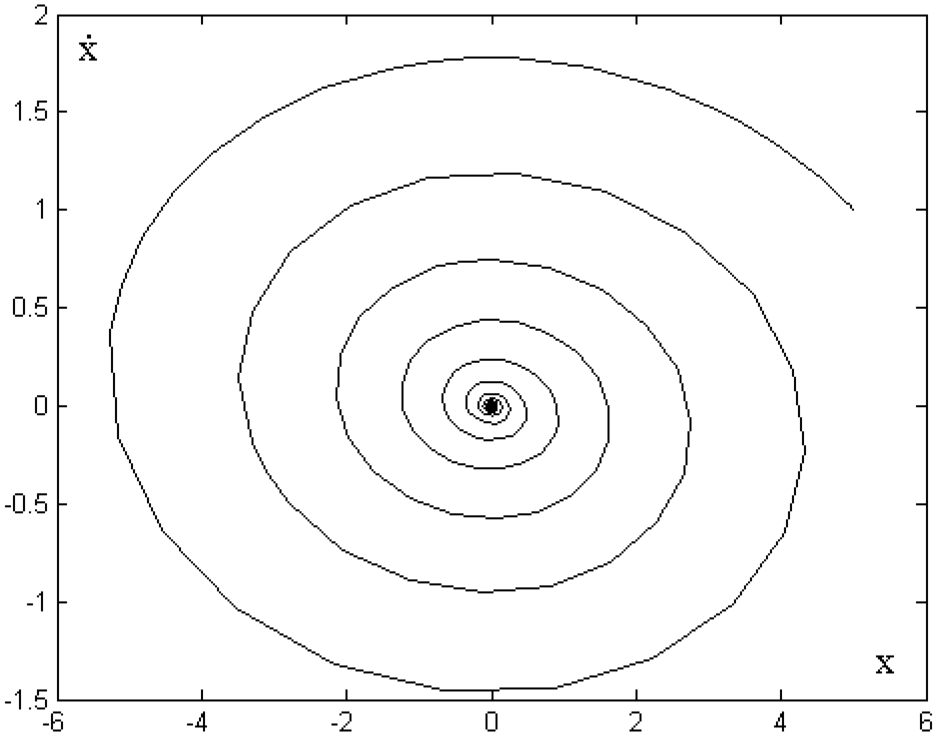

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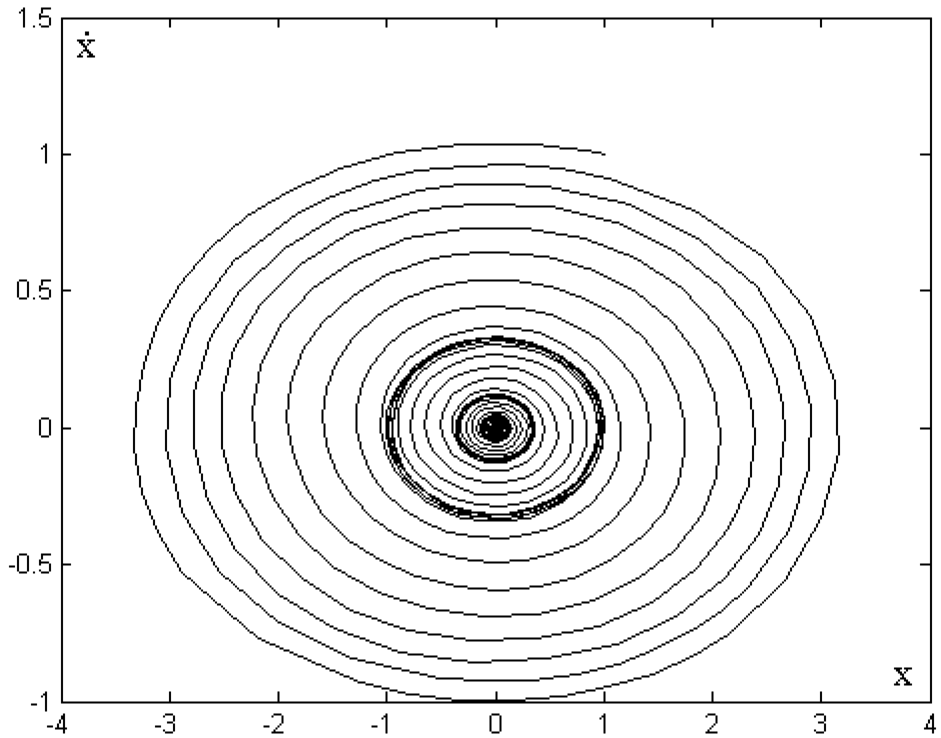

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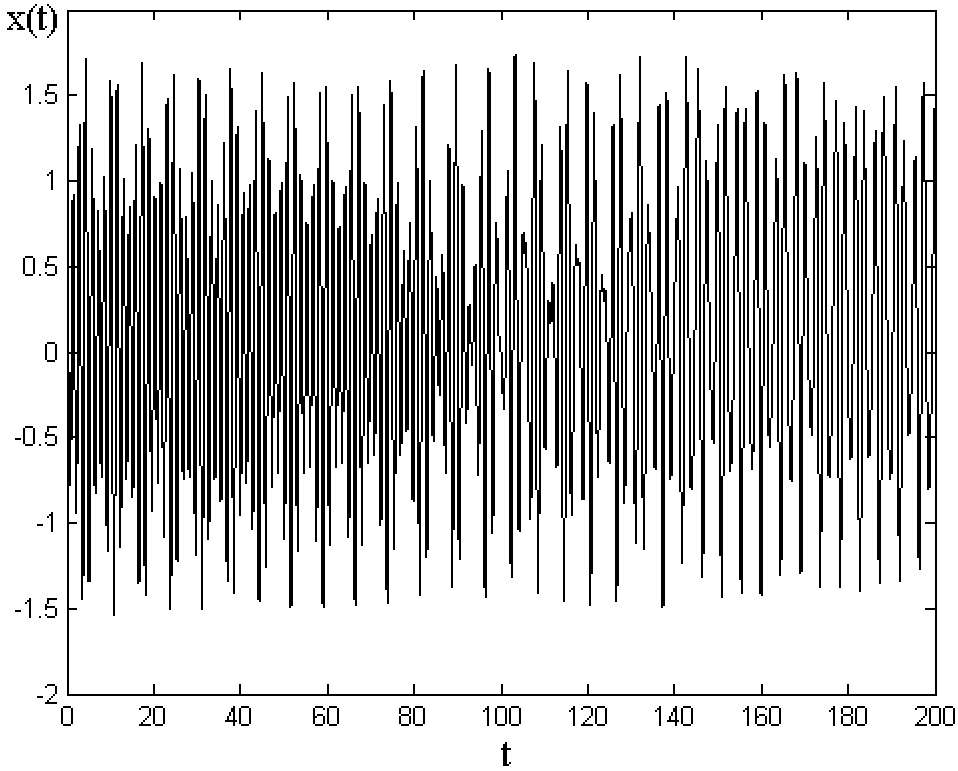
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0.5
0
-0.5
-1
-1.5
-2
0
20
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60
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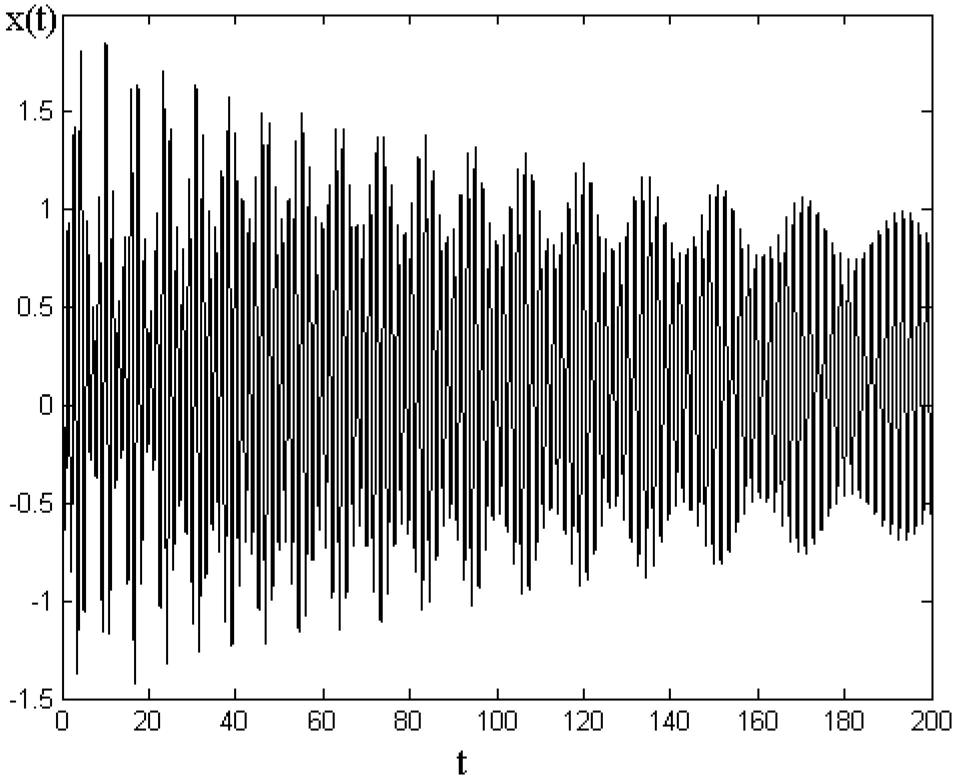
100
120
140
160
180
200
t



x(t)
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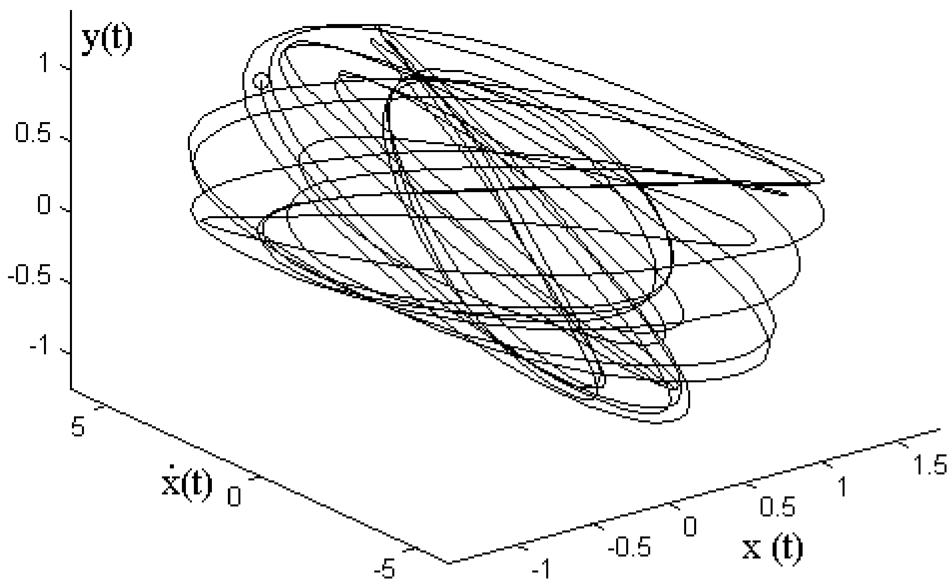

y(t)
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0.5
0
-0.5
-1
5
$\dot{x}$(t)
0
-5
-1
-0.5
0
0.5
1
1.5
x (t)